

**A SIMPLE COLLOCATION METHOD FOR
FITTING VISCOELASTIC MODELS TO
EXPERIMENTAL DATA**

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ABSTRACT

An easily applied collocation method is discussed for fitting the response of finite-element viscoelastic models to experimental stress-strain curves. It can be used with creep, relaxation, and steady-state oscillation data. The method is illustrated by means of two examples. As the first one, a model is obtained utilizing the dynamic shear compliance of polyisobutylene. In the second example we calculate a model from the tensile relaxation modulus of polymethyl methacrylate. With each case the model's response agreed with the experimental data within graphical accuracy over the entire frequency (or time) scale.

1. Introduction

It is well known that mechanical models consisting of springs and dashpots can be used to represent linear visco-elastic stress-strain behavior⁽¹⁾. Furthermore it is not necessary to use a general array of springs and dashpots, but only the simple arrangement of Voigt elements in series (Kelvin model) or Maxwell elements in parallel (Wiechert model)⁽²⁾. We shall not discuss the mechanics and mathematics of such models since reference (2) contains a relatively complete catalog of this information. Our objective here is only to present a simple method of calculating the parameters (spring and dashpot constants) in finite element models by fitting their response to experimental stress-strain data.

Other investigators have suggested various graphical and numerical schemes for obtaining model parameters. For example, Bland and Lee⁽³⁾ give a simple graphical scheme for fitting the response of four-element models to a narrow frequency band of steady-state sinusoidal ("dynamic") data. Welch⁽⁴⁾ also develops a graphical method for obtaining model parameters using dynamic data, however it is not limited to models with only four elements. Elder⁽⁵⁾ presents a least-squares technique, also for use with dynamic data, which leads to a set of non-linear algebraic equations for the model parameters.

In this paper we discuss a simple collocation method which can be used to closely fit the entire viscoelastic response curves

obtained in the standard tests of creep, relaxation, and steady-state oscillation. The method is presented by applying it to dynamic shear data obtained on glass-filled polyisobutylene and to the relaxation tensile modulus of polymethyl methacrylate. The stress-strain equations corresponding to the two models calculated in this paper are those used in reference (6) in two examples of visco-elastic stress analysis.

2. Property Representation of Polyisobutylene

Consider the problem of fitting the response of a model to the stress-strain data shown in Figure 1. This data was obtained on glass-filled polyisobutylene under constant frequency sinusoidal loading in shear. (7) The stress, $\sigma_0 e^{i\omega t}$, and strain, $\epsilon_0 e^{i\omega t}$, ($i = \sqrt{-1}$, ω = frequency given in radians per unit time) in such a test can be related by means of the complex compliance $J^*(\omega)$; thus

$$\frac{\epsilon_0 e^{i\omega t}}{\sigma_0 e^{i\omega t}} = \frac{\epsilon_0}{\sigma_0} = J^*(\omega) = J'(\omega) - i J''(\omega) \quad (1)$$

where $J'(\omega)$ and $J''(\omega)$ are real functions which are defined as the real and imaginary components of $J^*(\omega)$, respectively.

Let us now fit a Kelvin model to the data by collocating the model's response, $J_m^*(\omega) = J_m'(\omega) - i J_m''(\omega)$, with the experimental values. The response of the model is given by (2)

$$J_m'(\omega) = J_g + \sum_{i=1}^n \frac{J_i}{\omega^2 \tau_i^2 + 1} \quad (2a)$$

$$J_m''(\omega) = \sum_{i=1}^n \frac{\omega \tau_i J_i}{\omega^2 \tau_i^2 + 1} \quad (2b)$$

where J_g is the glassy (high frequency) compliance, J_i and τ_i are model constants which are to be evaluated by equating J_m' and J' at n values of frequency, ω_i , and choosing the τ_i such that $\omega_i \tau_i = 1$ ($i = 1, 2, \dots, n$)*. This procedure leads to a set of n linear algebraic equations for the constants J_i **. The way in which the τ_i are chosen is somewhat arbitrary; however by setting

* It may often be desirable to add another term to the series (2) in order to satisfy the condition

$$J_g + \sum_{i=1}^{n+1} J_i = J(\omega) \Big|_{\omega=0}$$

However, for the present example it turned out that this condition was closely satisfied without making it an explicit requirement.

** Of course, one could collocate at $2n$ points, say, in order to evaluate all $2n$ constants, J_i and τ_i . However, the resulting set of equations would be nonlinear and consequently quite difficult to solve.

$\tau_1 = 1/\omega_1$, each point of collocation corresponds to a frequency at which one of the terms in the series (2a) attains a value which is one half of its maximum ($\omega = 0$) value. In the present example this choice will be seen to produce a set of equations which are easily solved.

It should be observed that once the constants J_1 and τ_1 are known, the imaginary component J_m'' can be calculated from (2b) and compared with the data in Figure 2. If J_m' and J' are in good agreement then, apart from measurement errors, any significant difference between J_m'' and J'' can be attributed to nonlinearities or inertia effects.

By referring to Figure 1, it is seen that a good fit might be obtained if

$$\omega_l = 10^{(l-2)} \quad ; \quad (l=1, 2, \dots, 10) \quad (3)$$

so that the "retardation times" τ_l are chosen to be

$$\tau_l = 10^{(2-l)} \quad ; \quad (l=1, 2, \dots, 10) \quad (4)$$

Collocation at the values (3) yields the following system of linear equations for evaluation of the J_1 :

$$\begin{bmatrix}
 \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 & 1 & 1 & 1 \\
 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 & 1 & 1 \\
 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{101} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2}
 \end{bmatrix}
 \begin{bmatrix}
 J_1 \\
 J_2 \\
 J_3 \\
 J_4 \\
 J_5 \\
 J_6 \\
 J_7 \\
 J_8 \\
 J_9 \\
 J_{10}
 \end{bmatrix}
 =
 \begin{bmatrix}
 J'(10^4) - J_9 \\
 J'(1) - J_8 \\
 J'(10) - J_7 \\
 J'(10^2) - J_6 \\
 J'(10^3) - J_5 \\
 J'(10^4) - J_4 \\
 J'(10^5) - J_3 \\
 J'(10^6) - J_2 \\
 J'(10^7) - J_1 \\
 J'(10^8) - J_0
 \end{bmatrix}
 \quad (5)$$

where $J_5 = 3.16 \times 10^{-11}$. The elements of the matrix which are shown as unity or zero are not exactly these values, but are close enough to permit the approximation. Since the matrix is nearly triangular the equations can be easily solved by iteration as follows: Let the first estimation of the J_i be denoted as $J_i^{(1)}$. Solve for $J_1^{(1)}$ by neglecting all elements $1/101$ and $1/10^4 + 1$. This provides a set of 1 x 1 equations by starting with $J_{10}^{(1)}$, e.g.

$$J_{10}^{(1)} = 2 [J'(10^8) - J_9] \quad (6)$$

The second approximation $J_1^{(2)}$ is then obtained by retaining all

elements in (5), and letting the first solution $J_i^{(1)}$ multiply the elements which were initially neglected. For example, the $J_i^{(2)}$ are calculated from:

$$\begin{aligned}
 \frac{J_1^{(2)}}{2} + \frac{J_2^{(2)}}{1.01} + J_3^{(2)} + J_4^{(2)} + \dots + J_{10}^{(2)} &= J'(10^{-1}) - J_g \\
 \frac{J_1^{(1)}}{101} + \frac{J_2^{(2)}}{2} + \frac{J_3^{(2)}}{1.01} + J_4^{(2)} + \dots + J_{10}^{(2)} &= J'(1) - J_g \\
 \frac{J_1^{(1)}}{10^{4+1}} + \frac{J_2^{(1)}}{101} + \frac{J_3^{(2)}}{2} + \frac{J_4^{(2)}}{1.01} + \dots + J_{10}^{(2)} &= J'(10) - J_g \\
 \vdots & \\
 \text{etc.} &
 \end{aligned}
 \tag{7}$$

which can be solved as a set of 1×1 equations by starting with $J_{10}^{(2)}$. This iteration procedure is to be repeated until the desired accuracy is obtained. For the problem at hand it was found that J_m' and J' agreed within graphical accuracy by using the third approximation, which is

$$\begin{aligned}
 J_1 &= 357 \times 10^{-11} & J_6 &= 250 \times 10^{-11} \\
 J_2 &= 533 \times 10^{-11} & J_7 &= 80.8 \times 10^{-11} \\
 J_3 &= 3960 \times 10^{-11} & J_8 &= 22.2 \times 10^{-11} \\
 J_4 &= 3580 \times 10^{-11} & J_9 &= 4.00 \times 10^{-11} \\
 J_5 &= 1210 \times 10^{-11} & J_{10} &= 2.22 \times 10^{-11}
 \end{aligned}
 \tag{8}$$

It was mentioned above that (2b) can be used to predict the imaginary component of J^* , and hence serve as a check on the linearity assumption and the assumption that inertia is negligible. The predicted and experimental curves which are shown in Figure 2 agree well except at high frequencies.

As another point of interest, it is observed that the nearly triangular matrix (5) results from the fact that each term in the series (2a) is weakly dependent on frequency except for about 1 1/2 decades in the neighborhood of $\omega \approx 1/\tau_i$. Hence, for data in which it is necessary to collocate at points spaced closer than one decade, such simplicity does not occur and it may be expedient to use a high-speed computer in solving for the J_i .

Finally, the values (4) and (8) can be used to immediately write the equation relating Laplace transformed shear stress $\bar{\sigma}$, and transformed shear strain, $\bar{\epsilon}$, since⁽²⁾

$$\bar{\epsilon} = J(p) \bar{\sigma} \quad (9a)$$

where $J(p)$ is the operational shear compliance,

$$J(p) = J_g + \sum_{i=1}^{10} \frac{J_i}{\tau_i p + 1} \quad (9b)$$

and p is the Laplace transform parameter. The curve of $J(p)$ for $0 \leq p < \infty$, which is plotted in Figure 1, can be used directly with the stress analysis methods discussed in reference 6.

3. Property Representation of Polymethyl Methacrylate

As the second example, we shall fit the response of a Wiechert model to the tensile relaxation modulus $E_R(t)$ for polymethyl methacrylate⁽⁸⁾ which is shown in Figure 3. The relaxation modulus

$E_m(t)$ for the finite-element model is given by the series⁽²⁾

$$E_m = E_e + \sum_{i=1}^n E_i e^{-\frac{t}{\tau_i}} \quad (10)$$

where E_e is the equilibrium (rubbery) modulus given by

$$E_e = E_R(t) \Big|_{t \rightarrow \infty} \quad (11)$$

Similar to the previous example, we choose the values τ_i and then collocate E_m and $E_R(t)$ in order to evaluate the coefficients E_i . For the particular example considered it is desirable to take as one condition

$$E_m(0) = E(0) \quad (i=1) \quad (12)$$

and choose ten other collocation points

$$t_i = 10^{(i-3)} \quad ; \quad (i=2, 3, \dots, 11) \quad (13)$$

For the "relaxation times" τ_i , take $\tau_i = 2 t_i$ *

$$\tau_i = 2 \times 10^{(i-3)} \quad ; \quad (i=1, 2, 3, \dots, 11) \quad (14)$$

* These values of τ_i are taken because they locate each collocation point at a time in which one term in the series (10) has a value of about one-half its maximum value.

Collocation at $t = 0$ and the values (13) yields the linear system:

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0.007 & 0.607 & 0.951 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0.007 & 0.607 & 0.951 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0.007 & 0.607 & 0.951 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607
 \end{bmatrix}
 \begin{bmatrix}
 E_1 \\
 E_2 \\
 E_3 \\
 E_4 \\
 E_5 \\
 E_6 \\
 E_7 \\
 E_8 \\
 E_9 \\
 E_{10} \\
 E_{11}
 \end{bmatrix}
 =
 \begin{bmatrix}
 E_R(0) - E_e \\
 E_R(10^1) - E_e \\
 E_R(1) - E_e \\
 E_R(10) - E_e \\
 E_R(10^2) - E_e \\
 E_R(10^3) - E_e \\
 E_R(10^4) - E_e \\
 E_R(10^5) - E_e \\
 E_R(10^6) - E_e \\
 E_R(10^7) - E_e \\
 E_R(10^8) - E_e
 \end{bmatrix}
 \quad (15)$$

where $E_e = 2.24 \times 10^7$. Except for the top row, the elements in the matrix that are shown as zero or unity represent the exponential terms in the series (10) which, while not exactly zero or unity, are extremely close to these values. It is observed that if the 0.007 elements are neglected a triangular matrix results, which leads to a set of 1 x 1 equations for the E_i by starting with E_{11} . These equations are easily solved to find

$$\begin{aligned}
E_1 &= 194 \times 10^7 & E_7 &= 41 \times 10^7 \\
E_2 &= 283 \times 10^7 & E_8 &= 13.8 \times 10^7 \\
E_3 &= 554 \times 10^7 & E_9 &= 3.68 \times 10^7 \\
E_4 &= 602 \times 10^7 & E_{10} &= 0.79 \times 10^7 \\
E_5 &= 388 \times 10^7 & E_{11} &= 0.96 \times 10^7 \\
E_6 &= 156 \times 10^7
\end{aligned}
\tag{16}$$

which make $E_m(t)$ graphically indistinguishable from $E_R(t)$.

When it is necessary to use collocation spacing which is closer than one-decade, it may be necessary to use an iteration scheme similar to the one discussed in part I, or else use a high-speed computer.

The tensile stress-strain equation corresponding to (10) is given by ⁽²⁾

$$\bar{\sigma} = E(p) \bar{\epsilon} \tag{17}$$

where $\bar{\sigma}$ and $\bar{\epsilon}$ are the Laplace transformed stress and strain and $E(p)$ is the operational Young's modulus

$$E(p) = E_e + \sum_{i=1}^{11} \frac{p \tau_i E_i}{p \tau_i + 1} \tag{18}$$

which is plotted in Figure 3 for $0 \leq p < \infty$. As in the previous example, this curve can be used with the stress analysis techniques developed in reference 6.

REFERENCES

1. Biot, M. A.: Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena. *Journal of Applied Physics*, Vol. 25, 1954. pp. 1385 - 91.
2. Williams, M. L.; Blatz, P. J.; Schapery, R. A.: Fundamental Studies Relating to Systems Analysis of Solid Propellants. GALCIT SM 61-5, California Institute of Technology, February 1961.
3. Bland, D. R.; Lee, E. H.: On Fitting a Four Element Viscoelastic Model to Measured Complex Compliance Functions. Technical Report No. 13, NORD 11496, Brown University, June 1955.
4. Welch, J. T., Jr.: Empirical Determination of Some Mathematical Models in Viscoelasticity. ERD - 148/1. North Carolina State College, August 1961.
5. Elder, A. S.: Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland. (Personal Communication).
6. Schapery, R. A.: Approximate Methods of Transform Inversion for Viscoelastic Stress Analysis. GALCIT SM 62-2, California Institute of Technology, January 1962.
7. Landel, R. F.: The Dynamic Mechanical Properties of a Model Filled System: Polyisobutylene - Glass Beads. *Trans. Society of Rheology*, Vol. 2, 1958, pp. 53-75.
8. Lee, E. H.; Rogers, T. G.: Solution of Viscoelastic Stress Analysis Problems Using Measured Creep or Relaxation Functions. Interim Technical Report No. 1, Project No. 1892-E, Brown University, August 1961.

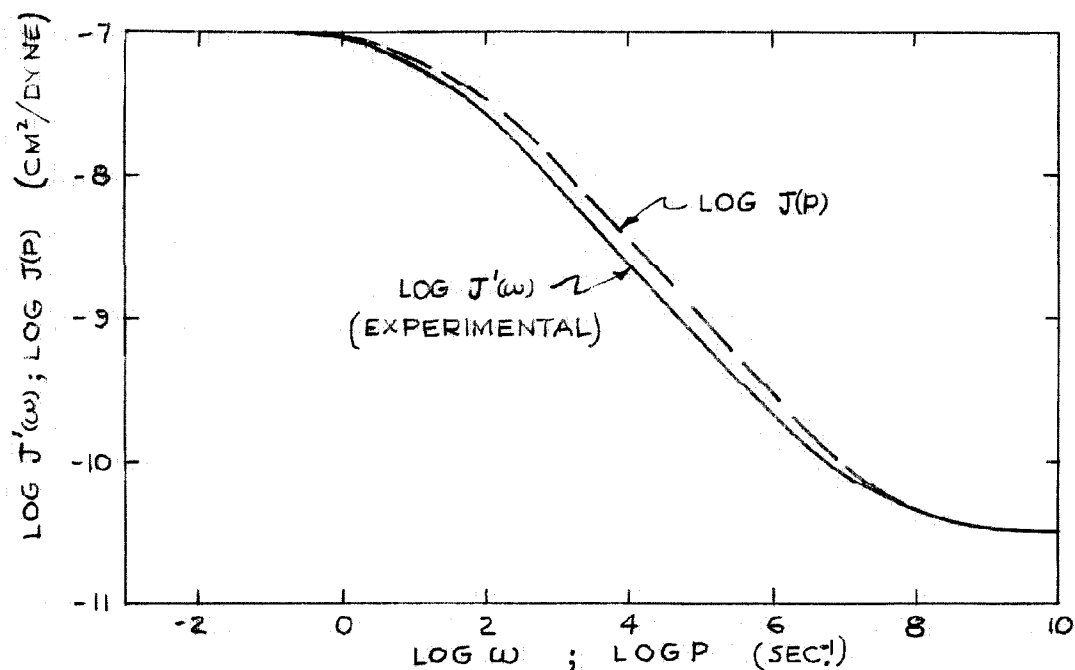


FIGURE 1. REAL PART OF COMPLEX COMPLIANCE AND OPERATIONAL COMPLIANCE FOR POLYISOBUTYLENE AT 12.5°C (IN SHEAR)

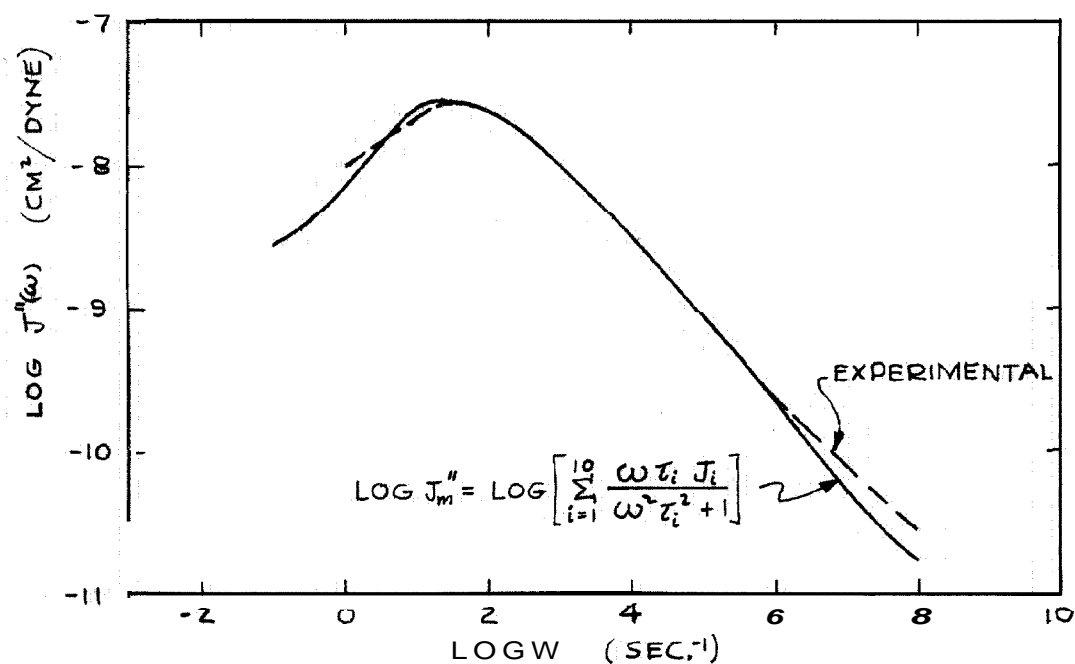


FIGURE 2. IMAGINARY PART OF COMPLEX COMPLIANCE FOR POLYISOBUTYLENE AT 12.5°C (IN SHEAR). COMPARISON OF EXPERIMENTAL AND MODEL RESPONSE.

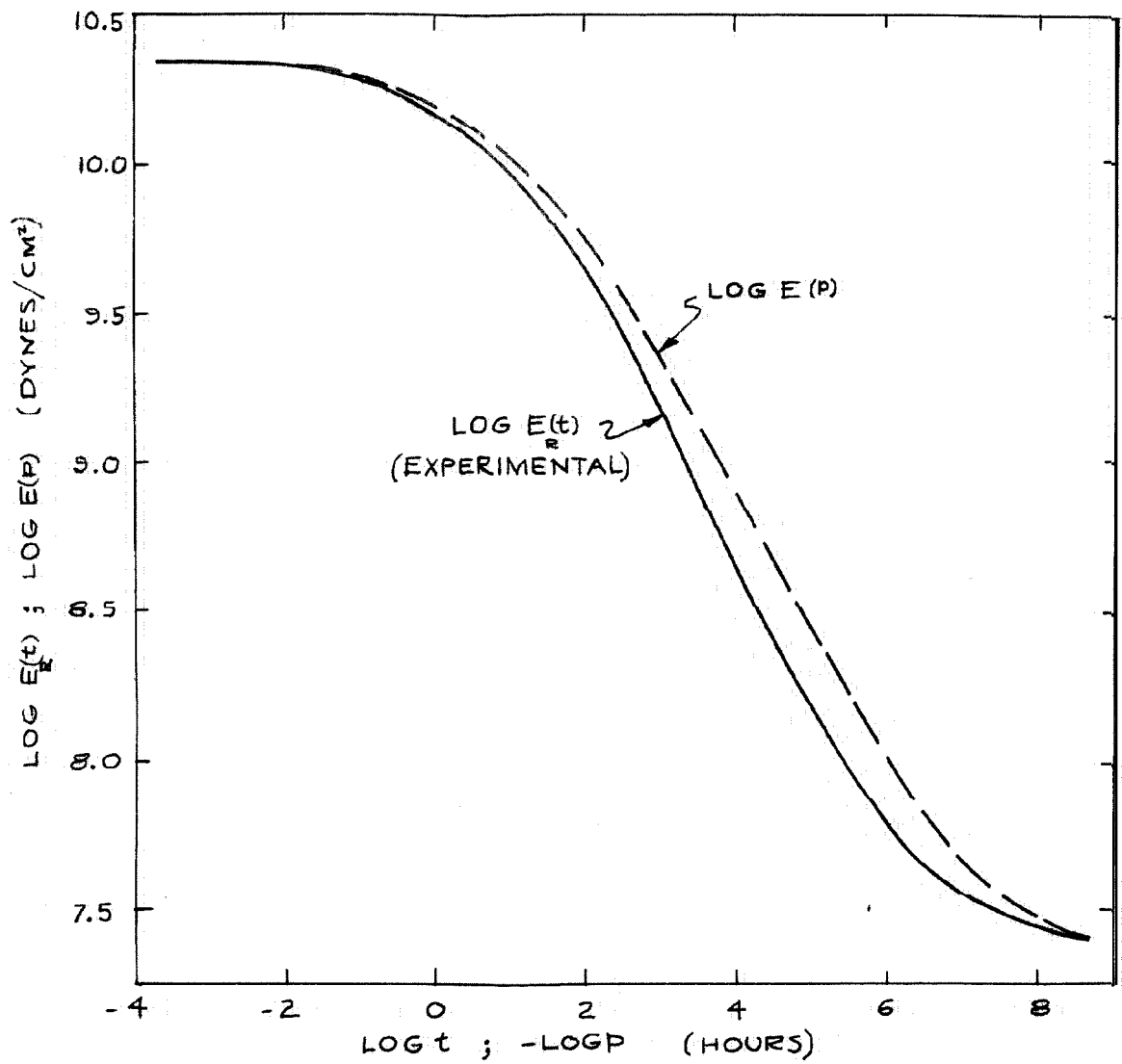


FIGURE 3. RELAXATION MODULUS AND OPERATIONAL MODULUS FOR POLYMETHYL METHACRYLATE AT 80°C (IN SIMPLE TENSION)